三. Integration on Path

1. Riemann Integration

$\varphi : [a, b] \mapsto \mathbb{C}$. It's a piecewise continuous function defined on the closed interval $[a, b]$. $[a, b]$ is a reel interval.

Then the Riemann Integral is

$$\int_a^b \varphi(t)\,dt = \int_a^b Re(\varphi(t))\,dt + j\int_a^b Im(\varphi(t))\,dt$$

2. Basic Properties

① Linearity

$$\int_a^b (\lambda\varphi(t) + \mu\psi(t))\,dt = \lambda\int_a^b \varphi(t)\,dt + \mu\int_a^b \psi(t)\,dt$$

② Cachy - Schwartz Inequality

$$\left|\int_a^b \varphi(t)\,dt\right| \leqslant \int_a^b |\varphi(t)|\,dt$$

③ $\varphi(x) = \varphi(a) + \int_a^x \varphi'(t)\,dt$ when $\varphi(x)$ has continuous derivative

3. Path

- If $\gamma(\cdot)$ defined in $\mathbb{C}$ can map $[a, b]$ to a C, and $\gamma(\cdot)$ is piecewise continuously differentiable, then $\gamma(\cdot)$ is called <u>path</u>.
- If $\gamma(a) = \gamma(b)$, the path is closed path.
- If we use γ^* to denote the range of γ, and $\gamma^* \subset S$. then γ is called a curve in S.

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The length of a curve γ is

$$L = \int_a^b |\gamma'(t)| \, dt$$

4. Integral on path.

$\gamma : [a,b] \mapsto \mathbb{C}$ is a path

$f(\cdot) : \gamma^* \mapsto \mathbb{C}$ is a continuous function.

Then the integral of $f(\cdot)$ on γ is

$$\int_\gamma f(z)\, dz = \int_a^b f(\gamma(t))\, \gamma'(t)\, dt$$

Theorem

$f(\cdot)$ is continuous on γ and $|f(\cdot)| \le M$, for points in γ^*

Length of γ is L, then

$$\left|\int_\gamma f(z)\, dz\right| \le ML$$

Proof $\left|\int_\gamma f(z)\, dz\right| \le \int_\gamma |f(z)| dz \le M \int_\gamma dz \le M \int_a^b |\gamma'(t)| dt = ML$

5. Fundamental Theorem for Integrals on Path.

If $F(\cdot)$ is the primitive of $f(\cdot)$, i.e. $F' = f$, then for any path $\gamma : [a,b] \mapsto \Omega$, we have

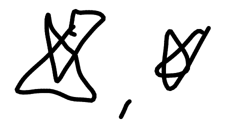
$$\int_\gamma f(t)\, dt = F(\gamma(b)) - F(\gamma(a))$$

So, we find that the result of integral in only related to the start and end point, rather than the detailed path.

6. Cauchy's Theorem for Triangles

If f is analytic on Ω and $T = [z_1, z_2, z_3, z_1]$ is any triangle s.t. $\mathrm{conv}(T) \subseteq \Omega$, Then $\int_T f(z)\,dz = 0$

7. Cauchy Thorem for Starlike Region

f is analytic on Ω, ()

f has primitive on Ω, then $\int_\gamma f(t)\,dt = 0$, γ is any closed path on Ω.

8. Converse Cauchy Theorem.

If $f: \Omega \mapsto \mathbb{C}$ is continuous and $\int_\gamma f(t)\,dt = 0$ for any closed path on Ω, then f has primitive on Ω.

Ex. γ is a circle centered at a, radius ρ. Prove that

$$\int_\gamma \frac{dz}{(z-a)^n} = \begin{cases} 2\pi i & \text{if } n=1 \\ 0 & \text{if } n \in \mathbb{Z},\ n \neq 1 \end{cases}$$

Note that, this is the core formula for Z-transformation.
The final result has nothing to do with a & ρ

Proof:

$z = \gamma(\theta) = \rho e^{i\theta} + a \quad \theta \in [0, 2\pi]$

When $n = 1$

$$\therefore \int_\gamma \frac{dz}{z-a} = \int_0^{2\pi} \frac{d(\rho e^{i\theta}+a)}{\rho e^{i\theta}} = \int_0^{2\pi} \frac{i\rho e^{i\theta}\, d\theta}{\rho e^{i\theta}} = 2\pi i$$

When $n \neq 1$, and n is integer

$$\int_\gamma \frac{dz}{(z-a)^n} = \int_0^{2\pi} \frac{i\rho e^{i\theta}\, d\theta}{(\rho e^{i\theta})^n} = i\rho^{-n+1} \int_0^{2\pi} e^{-i(n-1)\theta}\, d\theta = 0$$

The very useful application is that

$$\int_\gamma \frac{f(z)}{z-a}\, dz = 2\pi i f(a) \quad \text{if } a \text{ in } \partial\gamma \cup \operatorname{int}(\gamma)$$

9. Mean Value Theorem for analytic function

If $f(z)$ is analytic in $|z-a| < R$, and continuous in $|z-a| \leq R$

Then. $$f(a) = \frac{1}{2\pi} \int_0^{2\pi} f(a + Re^{i\varphi})\, d\varphi$$

Proof

$C: |z-a| = R \Rightarrow z = a + Re^{i\varphi} \ \forall \varphi$

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a}\, dz$$

$$= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(a+Re^{i\varphi})\, iRe^{i\varphi}\, d\varphi}{Re^{i\varphi}} = \frac{1}{2\pi} \int_0^{2\pi} f(a+Re^{i\varphi})\, d\varphi$$